SPRING 2024: BONUS PROBLEM 6

BP 6. Suppose A is a non-diagonalizable 3×3 matrix with $p_A(x) = (x-\lambda_1)^2(x-\lambda_2)$, so that $\dim(E_{\lambda_1}) = 1$. Thus, the JCF of A has the form $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$. Let v_2 be any vector satisfying $(A-\lambda_1 I)^2 v_2 = 0$, but v_2 is not an eigenvector for λ_1 and set $v_1 := (A-\lambda_1 I)v_2$. Take v_3 any eigenvector of λ_2. Show that v_1, v_2, v_3 are linearly independent.

Solution. To see that v_1, v_2, v_3 are linearly independent, suppose $av_1 + bv_2 + cv_3 = \vec{0}$. Apply the matrix $(A-\lambda_1 I)^2$ to both sides of this equation to get:

$$a(A-\lambda_1 I)^2 v_1 + b(A-\lambda_1 I)^2 v_2 + c(A-\lambda_1 I)^3 v_3 = \vec{0}.$$

By definition, $(A-\lambda_1 I)^2 v_2 = 0$. And $(A-\lambda_1 I)^2 v_1 = \vec{0}$, because we showed above that $(A-\lambda_1 I)v_1 = \vec{0}$. Thus, $c(A-\lambda_1 I)v_3 = \vec{0}$. However,

$$\begin{aligned}(A-\lambda_1 I)^2 v_3 &= (A-\lambda_1 I)\{(A-\lambda_1 I)v_3\} \\ &= (A-\lambda_1 I)\{Av_3 - \lambda_1 v_3\} = (A-\lambda_1 I)\{(\lambda_2-\lambda_1)v_3\} \\ &= (\lambda_2-\lambda_1)(A-\lambda_1 I)(v_3) = (\lambda_2-\lambda_1)(\lambda_2-\lambda_1)v_3.\end{aligned}$$

Thus, from $c(A-\lambda_1 I)v_3 = \vec{0}$, we have $c(\lambda_2-\lambda_1)^2 v_3 = 0$. Since $\lambda_2 - \lambda_1 \neq 0$, we must have $c = 0$. Thus, $av_1 + bv_2 = 0$. Now apply $A - \lambda_1 I$ to this equation. Since $(A-\lambda_1 I)v_1 = \vec{0}$ and $(A-\lambda_1)v_2 \neq \vec{0}$, we have $b(A-\lambda_1 I)v_2 = \vec{0}$ which implies $b = 0$. Finally, this leaves $av_1 = \vec{0}$, so $a = 0$. Therefore, v_1, v_2, v_3 are linearly independent.